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Entropy Production and Transports in a Conservative Multibaker Map with Energy

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For a previously introduced conservative multibaker map with energy, the Gaspard–Gilbert–Dorfman entropy production of the stationary state induced by the flux boundary condition is calculated and the entropy production is shown (i) to be nonnegative, (ii) to vanish in the fine-grained limit for finite chains, (iii) to take the phenomenologically expected value in the middle of the chain and to deviate from it near the boundaries, and (iv) to reduce to the phenomenological expression in the scaling limit where the lattice site $n \in \mathbf{Z}$ and time $t \in \mathbf{Z}$ are scaled respectively as $n = L_\xi X$ and $t = L_\tau T$ and the limits of $L_\xi \to +\infty$ and $L_\tau \to +\infty$ are taken while keeping the diffusion coefficient $D = lL_\tau/L_\xi^2$ constant, l being the transition rate of the model. The mass and heat transports are also studied in the scaling limit under an additional assumption that the edges of the chain are in equilibrium with different temperatures. In the linear heat transport regime, Fourier's law of heat conduction and the thermodynamic expression of the associated entropy production are obtained.

KEY WORDS: Nonequilibrium states; coarse-grained entropy production; singular measure; driven system; dynamical chaos; transport law.

1. INTRODUCTION

Understanding the microscopic entropy production is a long-standing problem in statistical mechanics.[1, 2] Recently, stimulated by progress in dynamical systems theory, the problem has been reinvestigated particularly for hyperbolic systems.[2–11] For this purpose, multibaker maps and their generalizations are used as prototypes of hyperbolic systems.[3, 6–13] Such an approach is expected to provide a useful information about many-body

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systems because of the Gallavotti–Cohen hypothesis,[14] which asserts that the microscopic dynamics of an N-body system is of hyperbolic character for sufficiently large numbers N of particles.

A multibaker map[3, 12, 13] is a lattice extension of the conventional baker transformation. It is a hyperbolic dynamical system, exhibits deterministic diffusion and has been used to study transport properties. While most of the proposed multibaker maps describe one kind of transports which is related to the probability conservation and is usually identified as the mass transport, several extensions are possible to describe two or more kinds of transports. In an extension by Mátyás, Tél and Vollmer,[10] the dynamics was not changed and a "temperature" field and a related "heat" flow were introduced in addition to the "mass" density. These authors have shown that their model describes "mass" and "heat" transports with cross effects and that the entropy production in their sense is consistent with nonequilibrium thermodynamics. On the other hand, in the conventional statistical mechanics, the whole system is considered to be conservative and the heat transport is related to the conservation of energy, which is a dynamical variable, but not a field defined on the phase, space. So, we extended a multibaker map by introducing a new coordinate corresponding to kinetic energy. Then, we required the multibaker map to be volume-preserving and time-reversal symmetric, and to conserve the sum of the "kinetic energy" and an "external potential energy."[9] Note that the "kinetic energy" is a mere phase-space coordinate of the map and the "external potential energy" is a mere phase-space field controlling the map parameters (i.e., the transition rates of the multibaker map). For the reasons to be explained in the next section, these quantities can be considered as the "kinetic" and "external potential" energies of our model, respectively.

In ref. 9, we showed that our model admits nonequilibrium stationary states carrying nonvanishing flows, which can be described by fractal distribution functions. On the basis of the aforementioned identifications, the mass and energy flows were calculated by using the fractal distribution functions and these flows were found to obey the phenomenological transport laws such as Fick's law. Moreover, in an appropriate scaling limit,[7, 15] [3] the coarse-grained entropy production in the sense of Gaspard[6] and

[3] In ref. 9, following Vollmer, Tél and Breymann (VTB),[7] we considered the scaling limit where the time step τ and lattice spacing d tend to zero while keeping the diffusion coefficient $D = l\,d^2/\tau$ constant, where l stands for the transition rate with no external field. In this paper, we consider the scaling limit in the sense of Bunimovich and Sinai (BS),[15] where the lattice site n and time t are scaled respectively as $n = L_\xi X$ and $t = L_\tau T$ and the limits $L_\xi \to +\infty$ and $L_\tau \to +\infty$ are taken while keeping $D = lL_\tau/L_\xi^2$ constant. The two scaling limits are mathematically equivalent via the replacements $\tau \leftrightarrow 1/L_\tau$ and $d \leftrightarrow 1/L_\xi$.

Gilbert–Dorfman[8] was investigated and shown to take an expression consistent with nonequilibrium thermodynamics.[9] However, in ref. 9, our treatment remained limited to a rough partition into relatively large cells and no spatial variation of the "temperature" was considered.

The aim of the present paper is to investigate in detail the cell-size dependence of the Gaspard–Gilbert–Dorfman entropy production for the nonequilibrium stationary states and to derive systematically the thermodynamic behaviors for mass and heat transports, including "temperature" variations.

The paper is organized as follows. In Section 2, we summarize the known results on the nonequilibrium stationary states of our multibaker model with energy. The entropy production is calculated in Section 3. The mass and heat transports are derived in Section 4. Conclusions are drawn in Section 5.

2. MULTIBAKER MAP WITH ENERGY

The multibaker map with energy introduced in ref. 9 is a caricature of the periodic Lorentz gas and was constructed on the basis of the following observation: For the two-dimensional ($2d$) periodic Lorentz gas without external field, the dimension of the phase space is four: two dimensions for the position and two for the momentum of the moving particle. Each trajectory of the particle can be fully determined by the coordinates of the particle at the successive collisions, which are given by the label of the scatterer, the scattering position θ of the particle on the scatterer, the direction ψ of the particle velocity just before the scattering and the magnitude of the particle velocity (or equivalently the kinetic energy of the particle). Without external field, the kinetic energy is conserved and can be omitted as dynamical variable. In this case, the dynamics of the Lorentz gas is controlled by a map defined on an array of (θ, ψ)-rectangles and this map resembles the usual multibaker map.[3]

Let us now consider the $2d$ Lorentz gas under an external field where the external potential is nearly constant on each scatterer. In this case, the trajectory can be determined by the coordinates at the successive collisions as before, but the kinetic energy of the particle is necessary to specify since the kinetic energy takes a different value when the particle hits a different scatterer, as a result of the external potential and the conservation of energy. Consequently, the dynamics is controlled by a map defined on an array of pillars, where the vertical direction is the kinetic-energy axis and the horizontal section represents a (θ, ψ)-rectangle at constant kinetic energy. The multibaker map with energy introduced in ref. 9 is designed in order to mimic this map.

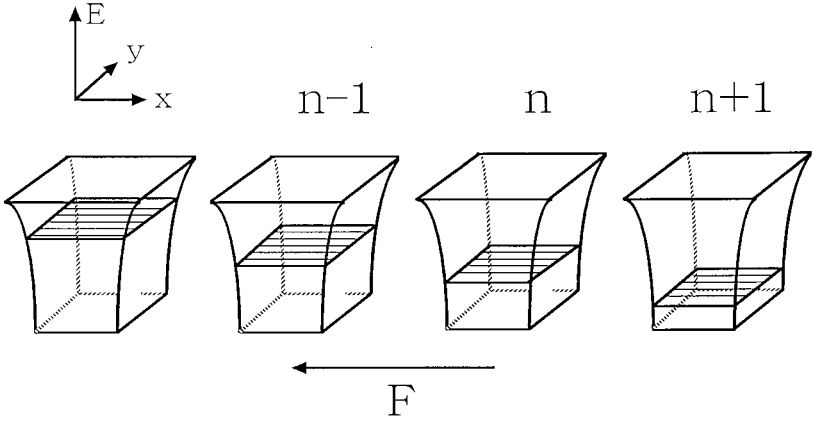

Fig. 1a. Schematic representation of the phase space Γ. The sectional area at "kinetic energy" E depends on E. The arrow represents the applied field and hatched squares corresponds to a constant total energy surface.

The phase space of our multibaker model is thus a chain Γ of three-dimensional cells as shown in Fig. 1(a):

$$\Gamma = \{(n, x, y, E) \mid n \in \mathbf{Z}, E \in \mathbf{R}^+, 0 \leqslant x \leqslant a(E), 0 \leqslant y \leqslant a(E)\} \qquad (1)$$

where $\mathbf{Z}$ and $\mathbf{R}^+$ stand for the sets of integers and of positive real numbers, respectively, and $a(E)$ is a positive function of E. In our analogy with the

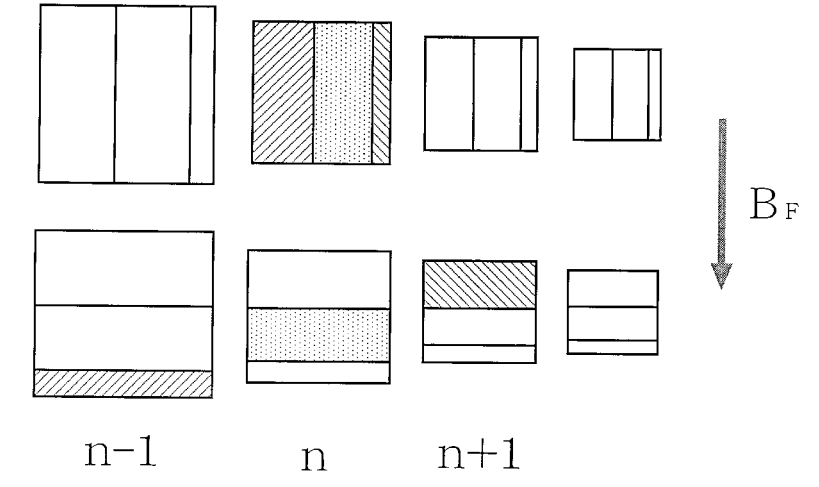

Fig. 1b. The multibaker map B_F on the constant total energy surface.

Lorentz gas under an external field, the variables n, x, y and E correspond respectively to the label of a scatterer, the angles θ and ψ (or ψ and θ) and the kinetic energy, and $a(E)^2$ represents the area of a section of each cell at constant "kinetic energy." From now on, the analogy with the Lorentz gas will no longer be used and we shall consider the possibility of a general E-dependence of the sectional area $a(E)^2$. As shown in ref. 9, the sectional area must have the form $a(E)^2 \propto e^{2E}$ in order for the map to satisfy the three following conditions: (i) conservation of the total energy, (ii) invertibility of the map on Γ and (iii) independence of the transition rates from the "kinetic" energy E.

In the case of a constant applied field F, the map B_F is given by [cf. Fig. 1b]

$$B_F(n, x, y, E) = \begin{cases} \left(n-1, \dfrac{x}{l^+ e^F}, l^+ e^F y, E+F\right), & 0 \leqslant \dfrac{x}{a(E)} \leqslant l^- \\ \left(n, \dfrac{x-l^- a(E)}{s}, sy + l^+ a(E), E\right), & l^- \leqslant \dfrac{x}{a(E)} \leqslant 1-l^+ \\ \left(n+1, \dfrac{x-(1-l^+)\,a(E)}{l^- e^{-F}}, \{l^- y + (1-l^-)\,a(E)\}\, e^{-F}, E-F\right), & \\ \qquad\qquad 1-l^+ \leqslant \dfrac{x}{a(E)} \leqslant 1 & \end{cases}$$

In these equations, $l^\pm \equiv 2l/(1+e^{\pm 2F})$ is the transition rate from the nth cell to the $(n \pm 1)$th cell and $s \equiv 1-2l$ is the self-transition rate, where $l \in (0, 1/2]$ is a real parameter.

We introduce the partially integrated distribution function G_t at fixed total energy E and time t which is evolved from the initial distribution ρ_0:

$$G_t(n, x, y, E) \equiv \int_0^y dy'\, \rho_0(B_\Phi^{-t}(n, x, y', E-Fn)) \tag{2}$$

Its evolution equation is derived from the definition of B_F. It is convenient to express G_t in terms of the rescaled coordinates $\xi \equiv x/a_n(E) \in [0, 1]$ and $\eta \equiv y/a_n(E) \in [0, 1]$ with $a_n(E) = ae^E e^{-Fn}$, which will be denoted as $\tilde{G}_t$: $\tilde{G}_t(n, \xi, \eta, E) \equiv G_t[n, \xi a_n(E), \eta a_n(E), E]$. Then, the evolution equation of the partially integrated distribution[(9)] reads as

$$\tilde{G}_{t+1}(n, \xi, \eta, E) = l^- e^{-F} \tilde{G}_t\left(n+1, l^- \xi, \frac{\eta}{l^+}, E\right) \tag{3}$$

for $\eta \in [0, l^+)$;

$$\tilde{G}_{t+1}(n, \xi, \eta, E) = l^- e^{-F} \tilde{G}_t(n+1, l^- \xi, 1, E) + s\tilde{G}_t\left(n, s\xi + l^-, \frac{\eta - l^+}{s}, E\right) \tag{4}$$

for $\eta \in [l^+, 1 - l^-)$; and

$$\tilde{G}_{t+1}(n, \xi, \eta, E) = l^- e^{-F} \tilde{G}_t(n+1, l^- \xi, 1, E) + s\tilde{G}_t(n, s\xi + l^-, 1, E) + l^+ e^{F} \tilde{G}_t\left(n-1, l^+ \xi + 1 - l^+, \frac{\eta - 1 + l^-}{l^-}, E\right) \tag{5}$$

for $\eta \in [1 - l^-, 1]$.

Under the flux boundary conditions

$$\begin{aligned} \tilde{G}_t(-1, \xi, \eta, E) &= \rho_-(E)\, a(E)\, e^F \eta \\ \tilde{G}_t(N+1, \xi, \eta, E) &= \rho_+(E)\, a(E)\, e^{-(N+1)F} \eta \end{aligned} \tag{6}$$

Eqs. (3)–(5) were shown to admit a unique stationary solution $\tilde{G}_\infty$ which does not depend on ξ[9]. The corresponding probability distribution $\Pi_{+\infty}(n, E)$ per energy and per site is

$$\Pi_{+\infty}(n, E) = a(E)^2 e^{-nF} \left\{ (\rho_+(E) - \rho_-(E)) \frac{e^{-(N+1)F} \sinh(n+1) F}{\sinh(N+2) F} + \rho_-(E)\, e^{-nF} \right\} \tag{7}$$

and it carries a nonvanishing probability flow $J_{n|n+1}(E)$:

$$\begin{aligned} J_{n|n+1}(E) &\equiv l^+ \Pi_{+\infty}(n, E) - l^- \Pi_{+\infty}(n+1, E) \\ &= -l(\rho_+(E) - \rho_-(E)) \frac{a(E)^2 e^{-NF} \tanh F}{\sinh(N+2) F} \end{aligned} \tag{8}$$

Note that the flow $J_{n|n+1}(E)$ from the nth site to the $(n+1)$th one does not depend on n because of the probability conservation. In terms of those quantities, we have

$$a_n(E)\, \tilde{G}_{+\infty}(n, \eta, E) = \Pi_{+\infty}(n, E)\, \eta - \frac{J_{n|n+1}(E)}{l} \varphi_n(\eta) \tag{9}$$

where φ_n is defined as the unique solution of the functional equation

$$\varphi_n(\eta) = \begin{cases} l^- \varphi_{n+1}\left(\dfrac{\eta}{l^+}\right) + \dfrac{l}{l^+}\eta, & 0 \leqslant \eta \leqslant l^+ \\ s\varphi_n\left(\dfrac{\eta - l^+}{s}\right) + l, & l^+ \leqslant \eta \leqslant 1 - l^- \\ l^+ \varphi_{n-1}\left(\dfrac{\eta - 1 + l^-}{l^-}\right) + \dfrac{l}{l^-}(1-\eta), & 1 - l^- \leqslant \eta \leqslant 1 \end{cases} \tag{10}$$

with the boundary conditions $\varphi_{-1}(\eta) = \varphi_{N+1}(\eta) = 0$.

3. ENTROPY PRODUCTION

As observed in ref. 6, because of the fractality of the nonequilibrium stationary states, the Gibbs entropy does not exist for infinitely large systems and the coarse-grained entropy would provide a microscopic entropy for open conservative systems. Here, in order to deal with a partition consisting of cells of different sizes, we adopt a slightly modified definition proposed by Gilbert and Dorfman.[8] However, contrary to ref. 8, we do not assume that the partition is generating and rather we follow the line of thoughts of ref. 6 for the following reason: As discussed in ref. 12, the multibaker map under the flux boundary condition (6) can be embedded into a conservative system of scattering type. Such a system is hyperbolic only on the fractal repeller,[3, 16] but not on the whole phase space because of the free motion. Accordingly, there is no generating partition covering the whole phase space. However, the measure associated with stationary state $\tilde{G}_{+\infty}$ is supported by the whole phase space. Hence, it is not appropriate to use the generating partition to calculate a coarse-grained entropy for the state $\tilde{G}_{+\infty}$.

For an arbitrary partition $\{D_j\}$ of the phase space, the coarse-grained entropy of a set A with respect to a measure ν associated with $\tilde{G}_{+\infty}$ is given by ref. 8:

$$S(A : \{D_j\}) \equiv \sum_{D_j \subset A} \nu(D_j) \ln \frac{\mu_0(D_j)}{\nu(D_j)} \tag{11}$$

where μ_0 is the reference Lebesgue measure and the summation is taken over all the cells D_j included in a set A. Because B_F preserves the measures

v and μ_0, the entropy production generated in one iteration of B_F is given by[6, 8]

$$\Delta_i S(A : \{D_j\}) = S(A : \{D_j\}) - S(A : \{B_F D_j\}) \tag{12}$$

In ref. 9, the entropy production of the kth cell C_k was calculated with the aid of a partition consisting of relatively large cells. Here we reinvestigate this calculation by using arbitrarily fine partitions and we extend our results.

We introduce a partition based on the cylindrical sets $\Sigma_{\omega_{-m},\ldots,\omega_l}(E)$:

$$\Sigma_{\omega_{-m},\ldots,\omega_l}(E) \equiv \bigcap_{j=-m}^{l} B_F^{-j} \Gamma_{\omega_j}(E) \tag{13}$$

where $\omega_{-m},\ldots,\omega_l \in \{0,\ldots,N\}$ and $\Gamma_n(E) \subset C_n$ is a slice at total energy E with width Δ_E:

$$\Gamma_n(E) = \{(n, x, y, E' - nF) \mid 0 \leqslant x \leqslant a_n(E'),\ 0 \leqslant y \leqslant a_n(E'),\ E \leqslant E' \leqslant E + \Delta_E\} \tag{14}$$

Because of the boundary condition, one has to take into account other cylindrical sets where at least one of the end symbols ω_{-m} and ω_l is -1 or $N+1$. In this regard, we call the nonempty cylindrical sets $\Sigma_{\omega_{-m},\ldots,\omega_l}(E)$ where all symbols $\omega_{-m},\ldots,\omega_l$ belong to $\{0,\ldots,N\}$ as type-I (m, l)-sets, those with $\omega_{-m} = -1$ or $N+1$ as type-II (m, l)-sets and those with $\omega_{-m} \in \{0, N\}$ and $\omega_l = -1$, $N+1$ as type-III (m, l)-sets. Note that, for type-II (m, l)-sets, $\omega_l = -1$, $N+1$ or $\omega_l \in \{0,\ldots,N\}$. Now, it is easy to show that a cylindrical set is a rectangle:

$$\Sigma_{\omega_{-m},\ldots,\omega_l}(E) = \left\{ (k, x, y, E' - nF) \;\middle|\; \frac{x}{a_k(E')} \in [\alpha_l, \beta_l],\ \frac{y}{a_k(E')} \in [\gamma_m, \delta_m],\ E \leqslant E' \leqslant E + \Delta_E \right\}$$

where the side lengths are

$$\begin{aligned} \beta_l - \alpha_l &= P_{k\omega_0} P_{\omega_0\omega_1} \cdots P_{\omega_{l-1}\omega_l} \\ \delta_m - \gamma_m &= e^{2F(k-\omega_{-m})} P_{\omega_{-m}\omega_{-m+1}} \cdots P_{\omega_{-2}\omega_{-1}} P_{\omega_{-1}k} \end{aligned} \tag{15}$$

and P_{nm} is the transition probability defined by $P_{n,n} = s$, $P_{n,n\pm1} = l^{\pm}$ and $P_{nm} = 0$ otherwise.

Now we consider the kth cell C_k which is a union of $\Gamma_k(E)$: $C_k = \bigcup_E \Gamma_k(E)$. It is obvious that, for given integers M and L, the constant-energy section $\Gamma_k(E)$ is decomposed as

$$\Gamma_k(E) = \bigcup_{\omega_{-M}\cdots\omega_L} \Sigma_{\omega_{-M}\cdots\omega_L}(E) \cup \bigcup_{m=1}^{M} \bigcup_{l=1}^{L} \left\{ \bigcup_{\omega_{-m}\cdots\omega_l}' \Sigma_{\omega_{-m}\cdots\omega_l}(E) \right\}$$
$$\cup \bigcup_{l=1}^{L} \left\{ \bigcup_{\omega_{-M}\cdots\omega_l}'' \Sigma_{\omega_{-M}\cdots\omega_l}(E) \right\}$$

where the first, second and third terms are unions of, respectively, the type-I (M, L)-sets, type-II (m, l)-sets with $m \leqslant M$, $l \leqslant L$ and type-III (M, l)-sets with $l \leqslant L$ contained in Γ_k (thus, $\omega_0 = k$). Note that, for the type-II sets, either $\omega_l = -1$, $N+1$ & $l < L$ or $-1 \leqslant \omega_l \leqslant N+1$ & $l = L$. From (15), the horizontal and vertical sides of the type-I cylindrical set are found to be smaller than $a_k(E)\,\tilde{l}^L$ and $a_k(E)\,\tilde{l}^M$, respectively, where $\tilde{l} \equiv \max(l^{\pm}, s)$. On the other hand, as schematically shown in Fig. 2, the type-II cylindrical sets may be larger and the type-III sets may be longer

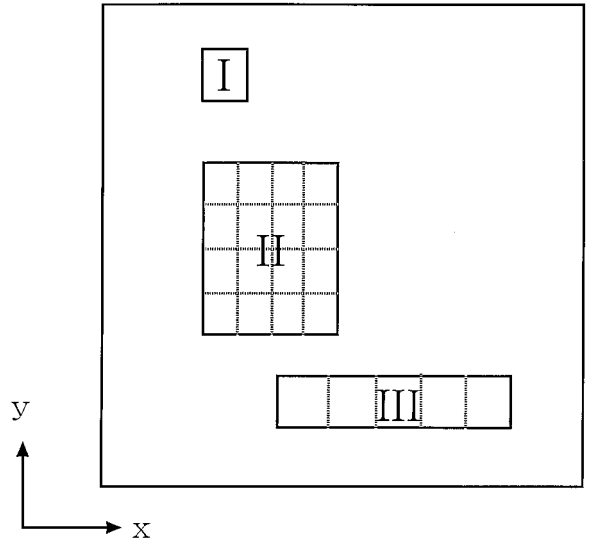

Fig. 2. Cylindrical sets contained in a constant energy surface at the kth cell. For given integers M and L, there exist three types of cylindrical sets. A type-I set corresponds to a symbolic sequence of the form $(\omega_{-M}, \omega_{-M+1}, ..., \omega_{L-1}, \omega_L)$ with $\omega_0 = k$, a type-II set to a sequence $(\omega_{-m}, ..., \omega_{l-1}, \omega_l)$ with $\omega_0 = k$ and $\omega_{-m} = -1$ or $N+1$ and a type-III set to a sequence $(\omega_{-M}, ..., \omega_{l-1}, \omega_l)$ with $\omega_0 = k$ and $\omega_l = -1$ or $N+1$. Dotted lines indicate subrectangles indexed by σ.

in the horizontal direction than the type-I sets. Hence, we further divide the type-II and type-III sets into subrectangles $\sum^{(\sigma)}_{\omega_{-m},\ldots,\omega_l}(E)$ smaller than the type-I sets (cf. Fig. 2). In this way, we obtain the desired partition:

$$\{D_j\} \equiv \bigcup_E \{\Sigma_{\omega_{-M}\cdots\omega_L}(E), \Sigma^{(\sigma)}_{\omega_{-m}\cdots\omega_l}(E)|_{\omega_{-m}=-1,N+1}, \Sigma^{(\sigma)}_{\omega_{-M}\cdots\omega_l}(E)|_{\omega_l=-1,N+1}\} \tag{16}$$

Then, the coarse-grained entropy (11) of the kth cell is given by

$$\begin{aligned} S(C_k:\{D_j\}) = &\sum_E \sum_{\omega_{-M}\cdots\omega_L} \nu(\Sigma_{\omega_{-M}\cdots\omega_L}(E)) \ln \frac{\mu_0(\Sigma_{\omega_{-M}\cdots\omega_L}(E))}{\nu(\Sigma_{\omega_{-M}\cdots\omega_L}(E))} \\ &+\sum_E \sum_{m=1}^{M}\sum_{l=1}^{L}\sum'_{\substack{\omega_{-m}\cdots\omega_l\\ \sigma}} \nu(\Sigma^{(\sigma)}_{\omega_{-m}\cdots\omega_l}(E)) \ln \frac{\mu_0(\Sigma^{(\sigma)}_{\omega_{-m}\cdots\omega_l}(E))}{\nu(\Sigma^{(\sigma)}_{\omega_{-m}\cdots\omega_l}(E))} \\ &+\sum_E \sum_{l=1}^{L}\sum''_{\substack{\omega_{-M}\cdots\omega_l\\ \sigma}} \nu(\Sigma^{(\sigma)}_{\omega_{-M}\cdots\omega_l}(E)) \ln \frac{\mu_0(\Sigma^{(\sigma)}_{\omega_{-M}\cdots\omega_l}(E))}{\nu(\Sigma^{(\sigma)}_{\omega_{-M}\cdots\omega_l}(E))} \end{aligned} \tag{17}$$

where the three terms are, respectively, the contributions from the type-I, type-II and type-III cylindrical sets and $\omega_0 = k$. Because of (15) and

$$\begin{aligned} &a_k(E)\{\tilde{G}_{+\infty}(k,\delta_m,E) - \tilde{G}_{+\infty}(k,\gamma_m,E)\} \\ &\quad = \Pi_{+\infty}(\omega_{-m},E)\, P_{\omega_{-m}\omega_{-m+1}}\cdots P_{\omega_{-1}k} \end{aligned} \tag{18}$$

$$\begin{aligned} &\mu_0(\Sigma_{\omega_{-m},\ldots,\omega_l}(E)) \\ &\quad = a_k(E)^2\,(\delta_m-\gamma_m)(\beta_l-\alpha_l)\,\Delta_E \end{aligned} \tag{19}$$

$$\begin{aligned} &\nu(\Sigma_{\omega_{-m},\ldots,\omega_l}(E)) \\ &\quad = a_k(E)\{\tilde{G}_{+\infty}(\omega_0,\delta_m,E) - \tilde{G}_{+\infty}(\omega_0,\gamma_m,E)\}(\beta_l-\alpha_l)\,\Delta_E \end{aligned} \tag{20}$$

in the limit of $\Delta_E \to 0$, the first term of (17) reduces to

$$\begin{aligned} &\sum_E \sum_{\omega_{-M}\cdots\omega_L} \nu(\Sigma_{\omega_{-M}\cdots\omega_L}(E)) \ln \frac{\mu_0(\Sigma_{\omega_{-M}\cdots\omega_L}(E))}{\nu(\Sigma_{\omega_{-M}\cdots\omega_L}(E))} \\ &\quad = \sum_E \sum_{\omega_{-M}\cdots\omega_L} \Pi_{+\infty}(\omega_{-M},E)\,\Delta_E P_{\omega_{-M}\omega_{-M+1}}\cdots P_{\omega_{-1}\omega_0}\cdots P_{\omega_{L-1}\omega_L} \\ &\qquad \times \ln \frac{a_{\omega_{-M}}(E)^2}{\Pi_{+\infty}(\omega_{-M},E)} \\ &\quad = \int dE \sum_{i,j=0}^{N} \Pi_{+\infty}(i,E)(P^M)_{i,k}\,(P^L)_{k,j} \ln \frac{a_i(E)^2}{\Pi_{+\infty}(i,E)} \end{aligned} \tag{21}$$

where P^M is the Mth power of the transition probability matrix $\{P_{mn}\}_{0 \leqslant m,\, n \leqslant N}$.

Since the measure ν has a constant density with respect to the Lebesgue measure μ_0 in the type-II sets, the ratio of two measures ν and μ_0 of a type-II subrectangle $\Sigma^{(\sigma)}_{\omega_{-m}\cdots\omega_l}$ is independent of the subrectangle index σ. Thus, the sum in the second term of (17) reduces to the sum over all type-II sets and can be calculated as the first term. This is also the case for the third term. We then obtain the entropy at the kth cell:

$$S(C_k : \{D_j\}) = \int dE \sum_{i=0}^{N} \Pi_{+\infty}(i, E)(P^M)_{i,k} \ln \frac{a_i(E)^2}{\Pi_{+\infty}(i, E)} + \sum_{m=1}^{M} \int dE \sum_{i=-1,\, N+1} \Pi_{+\infty}(i, E)(P^m)_{i,k} \ln \frac{a_i(E)^2}{\Pi_{+\infty}(i, E)} \quad (22)$$

Note that the final expression becomes independent of L thanks to the identity: $\sum_{j=0}^{N} (P^L)_{k,j} + \sum_{l=1}^{L} \sum_{j=-1,\, N+1} (P^l)_{k,j} = 1$.

Since $B_F \Sigma_{\omega_{-m}\cdots\omega_l}(E) = \Sigma_{\omega'_{-m-1}\cdots\omega'_{l-1}}(E)$ with $\omega'_s = \omega_{s+1}$, the partition $\{B_F D_j\}$ corresponds to the $(M+1, L-1)$-cylindrical sets. Hence, by substituting $M+1$ to M in (22), one obtains the entropy $S(C_k : \{B_F D_j\})$. Hence, the entropy production $\Delta_i S(C_k : \{D_j\})$ is

$$\Delta_i S(C_k : \{D_j\}) = S(C_k : \{D_j\}) - S(C_k : \{B_F D_j\}) = \int dE\, s_i(k, E) \quad (23)$$

where $s_i(k, E)$ is the entropy production per energy

$$s_i(k, E) = \sum_{i=0}^{N} \Pi_{+\infty}(i, E)(P^M)_{i,k} \ln \frac{a_i(E)^2}{\Pi_{+\infty}(i, E)} - \sum_{i=-1}^{N+1} \Pi_{+\infty}(i, E)(P^{M+1})_{i,k} \ln \frac{a_i(E)^2}{\Pi_{+\infty}(i, E)} \quad (24)$$

$$= -\sum_{i=0}^{N} \left\{ l^- \Pi_{+\infty}(i+1, E) \ln \frac{e^{-2F} \Pi_{+\infty}(i, E)}{\Pi_{+\infty}(i+1, E)} + l^+ \Pi_{+\infty}(i-1, E) \ln \frac{e^{2F} \Pi_{+\infty}(i, E)}{\Pi_{+\infty}(i-1, E)} \right\} (P^M)_{i,k} \quad (25)$$

In deriving (25), we used the recursion relation: $l^- \Pi_{+\infty}(i+1, E) + l^+ \Pi_{+\infty}(i-1, E) = 2l \Pi_{+\infty}(i, E)$.

Now we study the properties of $\Delta_i S(C_k : \{D_j\})$. Due to the recursion relation for $\Pi_{+\infty}(i, E)$ and the convexity of $\ln x$, the expression in the curly bracket of (25) satisfies

$$l^- \Pi_{+\infty}(i+1, E) \ln \frac{e^{-2F}\Pi_{+\infty}(i, E)}{\Pi_{+\infty}(i+1, E)} + l^+ \Pi_{+\infty}(i-1, E) \ln \frac{e^{2F}\Pi_{+\infty}(i, E)}{\Pi_{+\infty}(i-1, E)}$$

$$\leqslant 2l\Pi_{+\infty}(i, E) \ln 1 = 0$$

and, thus, the entropy production is non-negative: $\Delta_i S(C_k : \{D_j\}) = \int dE\, s_i(k, E) \geqslant 0$.

Secondly, since the eigenvalues of the matrix $\{P_{i,k}\}$ are $\kappa_j = [1 - 2l + 2\sqrt{l^+ l^-} \cos(\pi j/(N+2))]$ $(j = 1, ..., N+1)$ and $|\kappa_j| \leqslant |\kappa_1| < 1$, one has

$$|(P^M)_{i,k}| \leqslant K\, |\kappa_1|^M \to 0 \qquad (M \to \infty) \tag{26}$$

with some constant $K > 0$. Thus, the entropy production vanishes in the fine-grained limit:

$$\lim_{M \to \infty} \Delta_i S(C_k : \{D_j\}) = \int dE \lim_{M \to \infty} s_i(k, E) = 0 \tag{27}$$

Next, we consider the macroscopic limit, following ref. 6. Tracing back the argument used to derive Eq. (21) and using Eqs. (19), (20) and (9), we can rewrite the first term of (24) as

$$\sum_{i=0}^{N} \Pi_{+\infty}(i, E)(P^M)_{i,k} \ln \frac{a_i(E)^2}{\Pi_{+\infty}(i, E)}$$

$$= \sum_{\omega_{-M}, \ldots, \omega_{-1}} a_k(E)\{G_{+\infty}(k, \delta_M, E) - G_{+\infty}(k, \gamma_M, E)\}$$

$$\times \ln \frac{\delta_M - \gamma_M}{a_k(E)\{G_{+\infty}(k, \delta_M, E) - G_{+\infty}(k, \gamma_M, E)\}}$$

$$= -\sum_{\omega_{-M}, \ldots, \omega_{-1}} \left[\Pi_{+\infty}(k, E)\{\delta_M - \gamma_M\} - \frac{J_{k|k+1}}{l} \{\varphi_k(\delta_M) - \varphi_k(\gamma_M)\} \right]$$

$$\times \ln \left(\Pi_{+\infty}(k, E) - \frac{J_{k|k+1}}{l} \frac{\varphi_k(\delta_M) - \varphi_k(\gamma_M)}{\delta_M - \gamma_M} \right) \tag{28}$$

The second term can be calculated in a similar way. By expanding the resulting expression with respect to the ratio $J_{k\,|\,k+1}/(l\Pi_{+\infty})$ up to the second order and using (15) and the formula:

$$\varphi_k(\delta_m)-\varphi_k(\gamma_m)=\frac{l}{l^- - l^+}\,(1-e^{2F(k-\omega_{-m})})\,P_{\omega_{-m}\omega_{-m+1}}\cdots P_{\omega_{-1}k} \quad (29)$$

which follows from (10), we obtain

$$\begin{aligned} s_i(k,E) &= \frac{J_{k\,|\,k+1}^2}{2l^2\Pi_{+\infty}(k,E)} \sum_{\omega_{-M},\ldots,\omega_{-1}} \\ &\quad \times\left[\sum_{\omega_{-M-1}=-1}^{N+1} \frac{(\varphi_k(\delta_{M+1})-\varphi_k(\gamma_{M+1}))^2}{\delta_{M+1}-\gamma_{M+1}} - \frac{(\varphi_k(\delta_M)-\varphi_k(\gamma_M))^2}{\delta_M-\gamma_M}\right] \\ &\quad + \mathrm{O}\left(\frac{J_{k\,|\,k+1}^3}{l^2\Pi_{+\infty}^2}\right) \\ &= \frac{J_{k\,|\,k+1}^2\cosh^2 F}{l\Pi_{+\infty}(k,E)} \sum_{j=0}^{N} e^{2F(j-k)}(P^M)_{j,k} + \mathrm{O}\left(\frac{J_{k\,|\,k+1}^2}{l^2\Pi_{+\infty}^2}\right) \\ &= \frac{J_{k\,|\,k+1}^2}{l\Pi_{+\infty}(k,E)}\left\{1-\sum_{m=1}^{M}\sum_{j=-1,N+1}(P^m)_{j,k}\right\} \\ &\quad + \mathrm{O}\left(\frac{J_{k\,|\,k+1}^3}{l^2\Pi_{+\infty}^2}, \frac{FJ_{k\,|\,k+1}^2}{l\Pi_{+\infty}}\right) \end{aligned} \quad (30)$$

where we have used: $\sum_{j=0}^{N}(P^M)_{j,k}+\sum_{m=1}^{M}\sum_{j=-1,N+1}(P^m)_{j,k}=1$. Although the higher order terms look like of order of $J_{k\,|\,k+1}^3/\{l^3\Pi_{+\infty}^2\}$, they are actually of order of $J_{k\,|\,k+1}^3/\{l^2\Pi_{+\infty}^2\}$ because of an extra factor l from a φ_k-term. Since $P_{ij}\neq 0$ only for $|i-j|\leqslant 1$, we notice that $(P^m)_{-1,k}=(P^m)_{N+1,k}=0$ for all $m\leqslant M$, when the site k is sufficiently far from the boundary. Thus, the leading term of $s_i(k,E)$ takes the phenomenological expression $J_{k\,|\,k+1}^2/(l\Pi_{+\infty})$ in the middle part of the chain and deviates from it near the boundaries.[6, 8]

Finally, we consider the scaling limit in the sense of Bunimovich and Sinai (BS),[15] where the lattice site n and time t are scaled respectively as $n=L_\xi X$ and $t=L_\tau T$ and the limits $L_\xi\to+\infty$ and $L_\tau\to+\infty$ are taken while keeping the diffusion coefficient $D=lL_\tau/L_\xi^2$ constant. The field strength per unit distance $\mathscr{F}\equiv FL_\xi$, the probability flow per unit time per energy $j\equiv J_{k\,|\,k+1}L_\tau$, the probability density per energy $\pi_{+\infty}\equiv\Pi_{+\infty}L_\xi$ are of order unity and are smooth functions of $X\equiv k/L_\xi$. Note that, in ref. 9, we considered another scaling limit following Vollmer, Tél and Breymann

(VTB),[7] where the time step τ and lattice spacing d tend to zero while keeping $D = ld^2/\tau$ constant. The two scaling limits look different, but are mathematically equivalent via the replacements $L_\tau \leftrightarrow 1/\tau$ and $L_\xi \leftrightarrow 1/d$. Thus, the results of ref. 9 obtained in the VTB-scaling limit are valid also in the BS-scaling limit.

Accordingly, we have

$$\sigma_i(X, E) \equiv \lim_{\substack{L_\tau, L_\xi \to +\infty \\ D\text{: fixed}}} L_\tau L_\xi s_i(k, E) = \frac{j(X, E)^2}{D\pi_{+\infty}(X, E)} \tag{31}$$

where

$$j(X, E) = -D\left(2\mathscr{F}\pi_{+\infty}(X, E) + \frac{\partial \pi_{+\infty}(X, E)}{\partial X}\right) \tag{32}$$

Note that the scaling limit is meaningful only when $N \gg 1$ and $0 \ll k \ll N$.

4. TRANSPORTS

Now we study mass and heat transports in the scaling limit. Since, as discussed in Section 2, the new phase-space coordinate E can be regarded as a "kinetic energy," the argument E of $\pi_{+\infty}(X, E)$ and $j(X, E)$ can be regarded as the local "energy" of the particle. Thus, $p(X) \equiv \int dE\, \pi_{+\infty}(X, E)$, $\varepsilon(X) \equiv \int dE\, E\pi_{+\infty}(X, E)/p(X)$, $j_M(X) \equiv \int dE\, j(X, E)$ and $j_E(X) \equiv \int dE\, Ej(X, E)$ can be regarded, respectively, as the "mass density," the "energy per mass," the "mass flow" and the "energy flow." Then, as shown in ref. 9, we have

$$j_M(X) = -2D\mathscr{F}p(X) - D\frac{\partial p(X)}{\partial X} \tag{33}$$

$$j_q(X) = -Dp(X)\frac{\partial \varepsilon(X)}{\partial X} \tag{34}$$

where $j_q(X) \equiv j_E(X) - \varepsilon(X)\, j_M(X)$. Note that $p(X)$ and $\varepsilon(X)$ behave as independent fields for arbitrary $\mathscr{F}$ since $p(X)$ and $p(X)\,\varepsilon(X)$ depend only on $\int dE\, \rho^{\pm}$ and $\int dE\, E\rho^{\pm}$, respectively (for $\rho_{\pm}$, see Eq. (6)), and the quantities $\int dE\, \rho^{\pm}$ and $\int dE\, E\rho^{\pm}$ are mutually independent.

Now, the relation (33) can be interpreted as a phenomenological relation between the mass flow and the mass density, namely the first term of the right-hand side is a drift flow induced by the external field $\mathscr{F}$ and the second term represents a diffusion flow obeying Fick's law. Also, since

$j_E(X)$ represents the energy flow and $\varepsilon(X)\, j_M(X)$ corresponds to a energy flow associated with the mass flow, j_q of (34) can be interpreted as a "heat flow" and, thus, Eq. (34) as a phenomenological law of heat conduction. Now let us introduce here a spatially varying energy distribution: $h(X, E) \equiv \pi_{+\infty}(X, E)/p(X)$.[4] In terms of this energy distribution, Eqs. (31) and (32) give the following expression of the entropy production density $\tilde{\sigma}_i(X)$ per unit time

$$\tilde{\sigma}_i(X) \equiv \int dE\, \sigma_i(X, E) = \frac{j_M(X)^2}{Dp(X)} + Dp(X) \int \frac{dE}{h(X, E)} \left(\frac{\partial h(X, E)}{\partial X} \right)^2 \quad (35)$$

The first term agrees with the phenomenological expression due to the mass transport.[7, 10, 17] In short, we have obtained Eqs. (33), (34) and (35) which can be regarded as thermodynamical relations. In order to pursue the analogy with thermodynamics, we introduce a quantity corresponding to temperature. "Temperatures" at the left and right edges, T_- and T_+ respectively, are introduced by requiring the probability densities per energy at both edges to have the canonical form:

$$\pi_\pm(E) \equiv a(E)\, e^{-2n_\pm F} \rho_\pm(E)\, L_\xi = p_\pm \frac{e^{-E/T_\pm}}{z_\pm} \quad (36)$$

where $n_+ = N + 1$, $n_- = -1$, p_+ (p_-) is the mass density at the right (left) edge and $z_\alpha = \int dE \exp(-E/T_\alpha)$ ($\alpha = \pm$) are the partition functions. To derive relations corresponding to linear nonequilibrium thermodynamics, we assume that the temperature difference $\delta T_0 \equiv T_+ - T_-$ is small enough to retain the lowest order terms with respect to the temperature gradient.

In the lowest order in δT_0, $\pi_\pm(E)$ are

$$\pi_\pm(E) \simeq p_\pm \frac{e^{-E/T_0}}{z_0} \left\{ 1 \pm \frac{\delta T_0}{2T_0^2} (E - \langle E \rangle_0) \right\} \quad (37)$$

where $T_0 = (T_+ + T_-)/2$ is the average temperature, $z_0 = \int dE \exp(-E/T_0)$ and $\langle \cdots \rangle_0$ stands for the average with respect to the distribution $\exp(-E/T_0)/z_0$. Then, from (7), we have

$$\pi_{+\infty}(X, E) \simeq \frac{e^{-E/T_0}}{z_0} \left\{ p(X) + \frac{\delta T_0}{2T_0^2} \psi(X)(E - \langle E \rangle_0) \right\} \quad (38)$$

[4] In ref. 9, we supposed that this energy distribution does not vary in space: $h(X, E) = h(E)$.

where the function $\psi(X)$ is given by

$$\psi(X) \equiv (p_+ e^{\mathscr{F}\mathscr{L}} + p_- e^{-\mathscr{F}\mathscr{L}}) \frac{e^{-\mathscr{F}X} \sinh \mathscr{F}X}{\sinh \mathscr{F}\mathscr{L}} - p_- e^{-2\mathscr{F}X} \tag{39}$$

for a system of length $\mathscr{L} \equiv (N+1)/L_\xi$.

Now we introduce a function $T(X)$ corresponding to a temperature field by requiring that $h(X, E)$ is a local equilibrium distribution: $h(X, E) = \exp(-E/T(X))/z(X)$. In the lowest order in δT_0, Eq. (38) gives $T(X) = T_0 + \psi(X)\,\delta T_0/[2p(X)]$. And the energy per mass $\varepsilon(X)$ becomes

$$\varepsilon(X) = \frac{1}{p(X)} \int dE\, E\pi_{+\infty}(X, E) \simeq \langle E \rangle_0 + \frac{\langle \delta E^2 \rangle_0}{T_0^2} (T(X) - T_0) \tag{40}$$

where $\delta E = E - \langle E \rangle_0$. Hence, the heat flow $j_q(X)$ obeys

$$j_q(X) = -\lambda(X) \frac{\partial T(X)}{\partial X} \qquad \text{with} \quad \lambda(X) = Dp(X) \frac{\langle \delta E^2 \rangle_0}{T_0^2} \tag{41}$$

which can be interpreted as Fourier's law where $\lambda(X)$ is a (position-dependent) heat conductivity. On the other hand, with (38), Eq. (35) becomes

$$\tilde{\sigma}_i(X) = \frac{j_M(X)^2}{Dp(X)} + \frac{\lambda(X)}{T_0^2} \left(\frac{\partial T(X)}{\partial X} \right)^2 \tag{42}$$

which is the entropy production expected from thermodynamics.[10, 17] By comparing (33) and (41), we notice that there is no cross effects between mass and heat transports so that Onsager's relation holds trivially.

5. CONCLUSIONS

In the previous paper,[9] we introduced a conservative multibaker map with energy. In the scaling limit, a Smoluchowski-type equation was derived for a probability distribution function and the mass and energy flows were calculated. We also constructed the nonequilibrium stationary states under the flux boundary condition and we showed that the Gaspard–Gilbert–Dorfman entropy production with respect to a relatively coarse partition was non-negative and admitted an expression consistent with thermodynamics in the scaling limit. Here, as a continuation of the previous paper, we have shown that (i) the Gaspard–Gilbert–Dorfman entropy production with respect to a partition into cells of arbitrarily small size is non-negative for a stationary state induced by the flux boundary

condition, (ii) this entropy vanishes in the fine-grained limit for finite chains, (iii) it takes the phenomenologically expected value in the middle of the chain and a lower value near the boundaries up to the second order in $J_{k\,|\,k+1}$, and (iv) it reduces to the standard thermodynamic expression in the scaling limit in the sense of Bunimovich and Sinai.[15] Furthermore, the analogy with thermodynamics has been completed by introducing a variable corresponding to the temperature. By assuming, that the edges of the chain are in equilibrium with slightly different temperatures, we have derived a relation analogous to Fourier's law and we have recovered the thermodynamic expression of the associated entropy production. Hence, as in ref. 10, our model does describe mass and heat transports in a manner consistent with thermodynamics.

The results on the entropy production can be summarized as an equality with respect to the entropy production per energy $s_i(k, E)$:

$$\lim_{\varepsilon \to 0} \lim_{\substack{L_\tau, L_\xi \to +\infty \\ D\text{: fixed}}} \frac{L_\xi \Pi_{+\infty}}{(L_\tau J_{k\,|\,k+1})^2} \{L_\tau L_\xi s_i(k, E)\} = \frac{1}{D} > 0 \tag{43}$$

where the fine-grained limit $\varepsilon \to 0$ should be taken *after* the scaling limit. This is a generalization of the result obtained in ref. 6 and it implies that the fine-grained entropy production is an emerging property appearing in the scaling limit.

Here, we would like to emphasize the role of the fractality of the distribution for finding a positive entropy production. This fractality is essential for our conclusions that would not hold if the distribution was regular. Indeed, in Eq. (30), if the quotient $(\varphi_k(\eta) - \varphi_k(\eta'))/(\eta - \eta')$ was constant over the partitioning cell D_j, we would have the equality

$$\frac{\varphi_k(\delta_{M+1}) - \varphi_k(\gamma_{M+1})}{\delta_{M+1} - \gamma_{M+1}} = \frac{\varphi_k(\delta_M) - \varphi_k(\gamma_M)}{\delta_M - \gamma_M}$$

the sum in Eq. (30) would vanish because

$$\begin{aligned} &\sum_{\omega_{-M-1} = -1}^{N+1} \frac{(\varphi_k(\delta_{M+1}) - \varphi_k(\gamma_{M+1}))^2}{\delta_{M+1} - \gamma_{M+1}} - \frac{(\varphi_k(\delta_M) - \varphi_k(\gamma_M))^2}{\delta_M - \gamma_M} \\ &\quad = \frac{\varphi_k(\delta_M) - \varphi_k(\gamma_M)}{\delta_M - \gamma_M} \\ &\qquad \times \left[\sum_{\omega_{-M-1} = -1}^{N+1} \{\varphi_k(\delta_{M+1}) - \varphi_k(\gamma_{M+1})\} - \{\varphi_k(\delta_M) - \varphi_k(\gamma_M)\} \right] = 0 \end{aligned}$$

and, as a result, the entropy production would also vanish. Therefore, the fractality of the distribution is the very origin of the nonvanishing of the fine-grained entropy production. Note that this fractality of the stationary distribution appears in the scaling limit for area-preserving systems. For finite chains, the stationary state measure ν is very complicated and very different from a uniform measure but it is still absolutely continuous with respect to the Lebesgue measure so that the fine-grained entropy production vanishes.

Beside the nonequilibrium stationary states, the time-dependent properties of our model can also be studied such as the relaxation of the phase-space probabilities toward the nonequilibrium stationary state. With the aid of a method of ref. 18 (see also ref. 19), one can show that the distribution function $\tilde{G}_{+\infty}$ of the nonequilibrium stationary state given by Eq. (9) is approached when $t \to +\infty$ starting from the partially integrated distribution of any initial measure which is absolutely continuous with respect to the Lebesgue measure and possesses a density continuously differentiable in ξ, according to:

$$\tilde{G}_t(n, \xi, \eta, E) - \tilde{G}_{+\infty}(n, \eta, E) = \sum_{\substack{j=1 \\ |\kappa_j| > \lambda}}^{N+1} \kappa_j^t b_j(E)\, \gamma_j(n, \eta) + \delta\mathscr{G}_t(n, \xi, \eta, E) \tag{44}$$

where $\kappa_j = 1 - 2l + 2\sqrt{l^+ l^-}\cos(\pi j/(N+2))\,(<1)$, $\lambda = \max(1-2l, \sqrt{l^+ l^-})$, the j-sum runs over all $j = 1, \ldots, N+1$ satisfying $|\kappa_j| > \lambda$, and the function $\delta\mathscr{G}_t$ decays as $|\delta\mathscr{G}_t| = \mathrm{O}(t^2\lambda^t)$ uniformly with respect to n, ξ and η. In Eq. (44), the function $\gamma_j(n, \eta)$ is defined as the unique solution of the functional equation

$$\gamma_j(n, \eta) = \begin{cases} \dfrac{\sqrt{l^+ l^-}}{\kappa_j}\,\gamma_j\left(n+1, \dfrac{\eta}{l^+}\right), & 0 \leqslant \eta \leqslant l^+ \\ \dfrac{s}{\kappa_j}\,\gamma_j\left(n, \dfrac{\eta - l^+}{s}\right) + \dfrac{\sqrt{l^+ l^-}}{\kappa_j}\sin\left(\dfrac{(n+2)\,\pi j}{N+2}\right), & l^+ \leqslant \eta \leqslant 1 - l^- \\ \dfrac{\sqrt{l^+ l^-}}{\kappa_j}\,\gamma_j\left(n-1, \dfrac{\eta - 1 + l^-}{l^-}\right) & \\ \quad + \dfrac{s}{\kappa_j}\sin\left(\dfrac{(n+1)\,\pi j}{N+2}\right) + \dfrac{\sqrt{l^+ l^-}}{\kappa_j}\sin\left(\dfrac{(n+2)\,\pi j}{N+2}\right), & 1 - l^- \leqslant \eta \leqslant 1 \end{cases} \tag{45}$$

and the coefficient $b_j(E)$ is given by

$$b_j(E) = \frac{-2}{N+2} \sum_{n=0}^{N} \int_0^1 d\gamma_j(n, 1-\xi)\{\tilde{G}_0(n, \xi, 1, E) - \tilde{G}_{+\infty}(n, 1, E)\} \qquad (46)$$

in terms of the initial distribution function $\tilde{G}_0$. Thus, for any partitioning cell D_j, its measure $\nu_t(D_j)$ at time t converges to $\nu(D_j)$ as $t \to \infty$. Therefore, the total coarse-grained entropy of the kth cell converges toward its value (11) at the stationary state ν with partition-independent rates, as far as the system size N is finite and the sizes of the partitioning cells are nonvanishing. Accordingly, Eq. (44) shows that the long-time relaxation of the entropy and of its production toward their stationary values is controlled by the decay rate $-\ln \kappa_1$ which depends on the fundamental parameters of the system but not on the partition. Note that the scaling limit is not necessary to obtain the result (44) and that the coarse-grained entropy production of the stationary state (24) is positive even for finite N provided the partitioning cells are not too small. The same is true for the multibaker maps under the flux boundary condition studied by Gilbert and Dorfman.[8] We end by the remark that the extension of the present study of entropy production to nonstationary states could be carried out along the line of thoughts by Prigogine–Misra–Courbage[20] who introduced a monotonically decreasing negative-entropy-like quantity for Kolmogorov systems (which include the baker map), or along the one by Nicolis and coworkers[21] who proved a H-theorem for a coarse-grained entropy based on system-specific Markov partitions. This and the extension to continuous-time dynamical systems will be studied elsewhere.

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